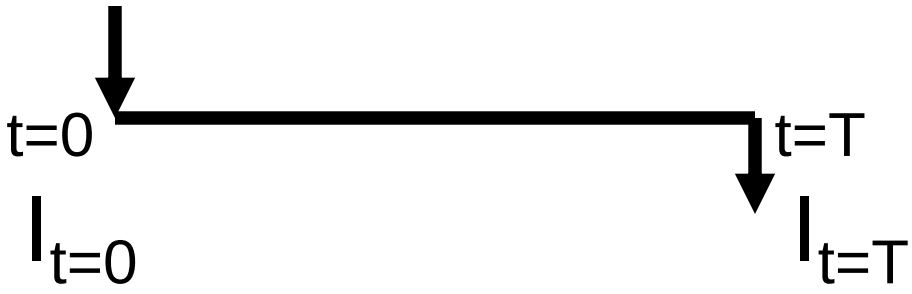
# Definition

$$r = (I_{t=T} - I_{t=0})/I_{t=o}$$



# Definition

Total return is calculated as:

$$R = I_{t=T}/I_{t=o}$$

Where:

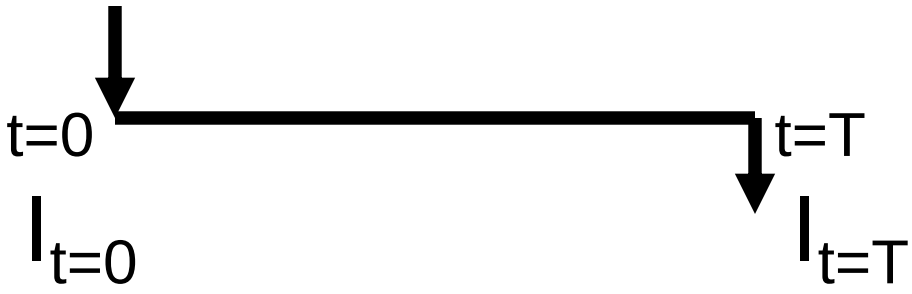
R = total return

$I_{t=T}$ = end of period (i.e., at time ′t=T′) wealth

$I_{t=0}$ = beginning of period (i.e., at time ′t=0′) wealth

# Definition

$$R = I_{t=T}/I_{t=o}$$



# Definition

Treasury bill (T-bill)

This is a government bond usually for 91 days, such that making an investment at the beginning of a period we can get our investment plus some interest back (depending on what is the rate of interest for the treasury bill). The rate of return for the T-bill is also termed as the risk free interest rate ($r_f$), as technically the investment for T-bill does not have any sort of risk or uncertainty involved with it.

# Definition

Long term bonds

This involves loaning money for a long duration of time. The investor (lender) buys the bonds (hence lends the money) to the borrower, i.e., the person who issues the bond for a long duration of time (say 2, 5, 10 years). In doing so the borrower commits to make cash payments at equal interval of times. These payments are known as coupon payments. We also have long term government bonds.

# Definition

Financial intermediaries

Financial intermediaries or Financial Institutions are organizations which issue financial claims against themselves. Let us consider an example. A typical commercial bank issues financial claims against itself in the form of debt (saving account, etc.)

# Definition

Return and Risk and diversification

A question which arises naturally while trying to analyze any stock is, how do we rank them according to a agreed upon metric. The general agreed upon metrics are (i) return; (ii) risk (variance). Other metrics like skewness and kurtosis can also be used.

# Definition

For an outcome if we are assured (100% sure) that it is sure to give us a benefit, then we are not concerned about the chance that it will not happen and the corresponding probability that we will be denied any benefit from that outcome. But in reality we know that in maximum of the cases we have to face the game of chance and hence be aware of the probabilistic return of the outcome.

# Definition

For an outcome, which is random or probabilistic we denote it by X. Corresponding to this r.v. we have an average/mean of this outcome, which is termed as the expected value, denoted by E[X]. Simultaneously along with the average return we are also concerned about the uncertainty involved in the outcome. This uncertainty we face regarding the outcome is the variance, denoted by V[X]. The variance is also known as the risk and it is some sort of cost associated with the uncertainty of the r.v.

# Definition

As just discussed risk is usually denoted by the variance, $\sigma^2=E[X - E(X)]^2$. There are other quantifiable ways of denoting this risk but for all practical purposes the second moment of a r.v. suffices to quantify this risk. Other measures of risk which we have just mentioned, like skewness and kurtosis are found out by using the third and fourth moment respectively, i.e., $E[X - E(X)]^3$ and $E[X - E(X)]^4$

# Definition

If we have X as a discrete variable then we have

$$E[X] = \sum_{\forall i} x_i f(x_i) \qquad V[X] = \sum_{\forall i} \{x_i - E[X]\}^2 f(x_i)$$

If we have X as a continuous variable then we have

$$E[X] = \int_{-\infty}^{+\infty} x f(x) dx \qquad V[X] = \int_{-\infty}^{+\infty} \{x - E[X]\}^2 f(x) dx$$

# Definition

For ′n′ assets denoted by $X_i$, we have their expected returns denoted by $\mu_i = E[X_i]$ and variances by $V[X] = \sigma^2_i$, respectively. Moreover if the correlation coefficient between the $i^{th}$ and the $j^{th}$ asset is denoted by $\rho_{ij}$, then the return and variance of the collection of assets (i.e., the portfolio) is given by the formulae given below, provided we have $w_i$ as the percentage of money invested in asset i.

# Definition

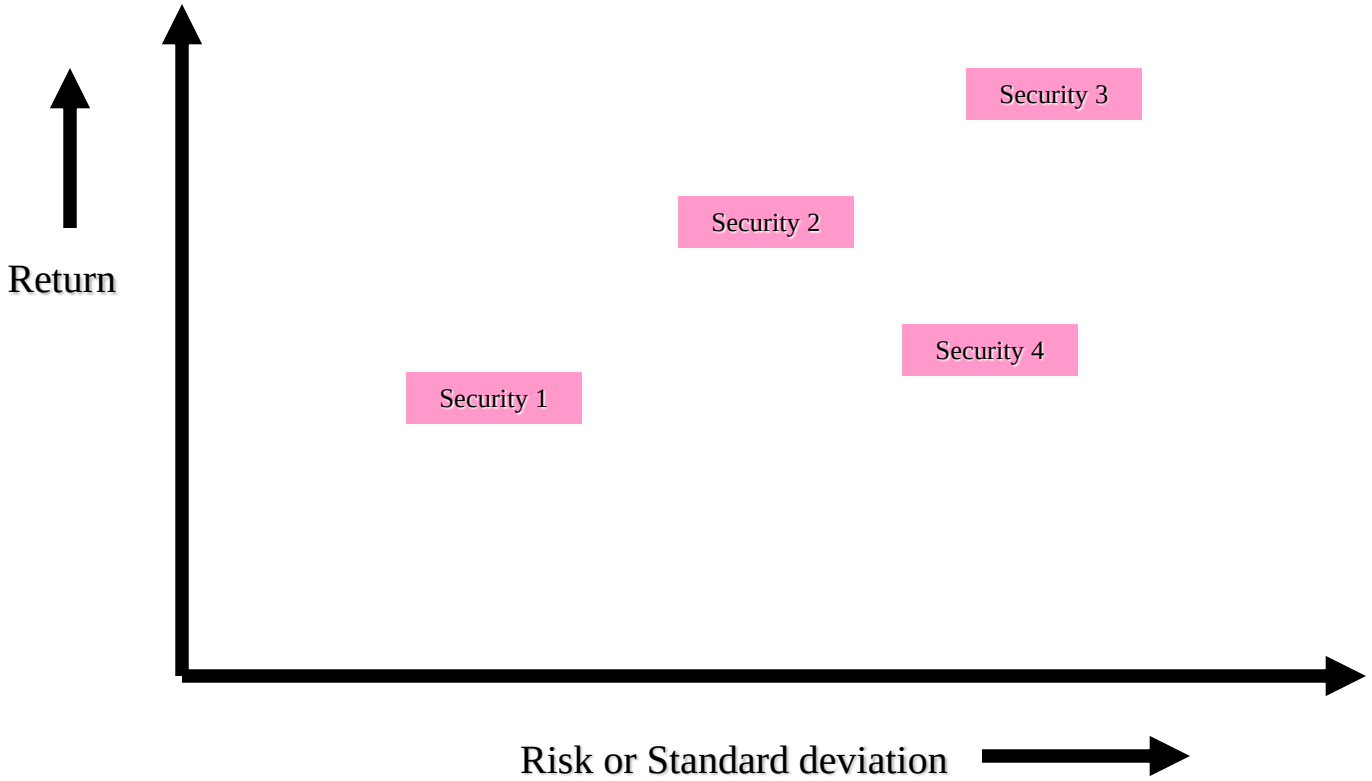
$$E[P] = \sum_{i=1}^{n} \bar{X}_i w_i$$

$$V[P] = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_{i,j} = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \rho_{i,j} \sigma_i \sigma_j$$

Also remember that

$$\sum_{i=1}^{n} w_i = 1$$

# Definition


Return
Security 3
Security 2
Security 4
Security 1
Risk or Standard deviation

# Investment Process

Short Sales (SS): Sometimes it is possible to sell an asset that you do not own though the process of short selling, or shorting, the asset. To do this you borrow the asset from someone who actually owns it. You then sell the borrowed asset to someone else for an amount $X_0$. Now at a later date in future, you repay by purchasing the asset at say an amount $X_1$ and return the asset to the lender, from whom you had borrowed. If $X_1 < X_0$, then you make a profit of amount $X_0 - X_1$, else you make a loss of amount $X_1 - X_0$. SS is considered risky and is even prohibited and discouraged within certain financial institutions as the losses can be unlimited if $X_1$ increases arbitrarily. In certain situations SS is allowed and the mechanism for allowing it is through some sort of ′caution deposit′ or ′security deposit′, about which we will discuss afterwards. Remember if the stock which you have taken on a loan for SS pays dividends during the period that you have borrowed it, then you must pay the same dividends to the person from whom you have borrowed the stock.

# Investment Process

Let us now determine the return associated with SS. We receive $X_0$ initially and pay $X_1$ later. So the $R=(X_1/X_0)$. Generally the returns of any asset is random, in the sense we do not know what would be the amount of money we should pay (receive) for buying (selling) the asset. Hence rather than working with the random variable (r.v), we are more interested in finding the average/expected return of the asset and as well as the variance of the returns. This is the notion which we will apply for calculating the portfolio return, where by a portfolio we will always denote a conglomeration of ′n′ number of assets, each of which has a certain random return.

# Investment Process

1) Set investment policy.
2) Perform security analysis.
3) Construct a portfolio.
4) Revise the portfolio.
5) Evaluate the performance of the portfolio.

# Investment Process

Set investment policy: Involves determining the investor's objectives and the amount of money he/she has for his/her investment.

Perform security analysis: In this step or process one examines several individual securities so that we know the fundamental characteristics of the assets. Under this we have two important concept which are:

- Technical Analysis
- Fundamental Analysis

# Investment Process

Construct a portfolio: This step involves identifying those specific assets to invest in and the amount of proportion of wealth to invest in each one of those assets. Under this the important concept of diversification, selectivity and timing will be discussed.

Revise the portfolio: This means periodic review of the portfolio and trying to find out how the portfolio is performing vis-à-vis the goal we have set for its evaluation. Under these step we need to re-consider some assets and include some and exclude some assets.

# Investment Process

Evaluate the performance of the portfolio: Under this step we periodically try to find out how our portfolio is performing and have proper benchmarks for its evaluation.

# Investment Process

**IMPORTANT**:

We will only discuss the portfolio theory as applied to managing financial assets. According to Markowitz's theory we know that riskness can be characterized by the variance. While the returns are characterized by the expected value. Utilizing these important concepts, we can formulate the general form of the optimization problem. About this we will discuss later.

Note: It will be worthwhile and quite useful to mention here that the **mean variance** analysis forms the basis for the Capital Asset Pricing Model (CAPM).

# Investment Process

When making an investment, the initial outlay of capital is known, but the amount to be returned is uncertain. Moreover this investment is considered for the case of a single investment period, i.e., the amount of money which is considered to be known is invested at the initial time period, say t = 0 and the payoff, which may be random is attained at the end of the period, i.e., t = T.

Examples for single period investment are:

- An investment in a zero coupon bond.
- An investment in a physical project that will not provide any payment until it is completed.

# Investment Process

We already know an asset is an instrument or a financial investment that can be brought and sold frequently. Examples can shares, bonds, land, gold, cash, etc. Suppose that a person purchases as asset at time zero, i.e., t = 0 and t = T period later he/she sells the asset. Then by definition the total return (R) for the investment is defined by:

# Investment Process

$$R = AR/AI$$

where:
R = total return
AR = amount received
AI = amount investment
Similarly rate of return (r) is defined as

$$r = (AR-AI)/AI$$

# Investment Process

If we denote AI = $X_0$ and AR = $X_1$ then:

$$R = 1 + r$$

and

$$X_1 = (1 + r) X_0$$

# Investment Process

Note, both the total return and the rate of return of a portfolio are equal to the weighted sum of the corresponding individual asset's total return and rate of return respectively. The weight of an asset being its relative weight.

Thus:

$$R_P = \sum_{i=1}^{n} w_i R_i \qquad r_P = \sum_{i=1}^{n} w_i r_i$$

and

$$\bar{R}_P = \sum_{i=1}^{n} w_i \bar{R}_i \qquad \bar{r}_P = \sum_{i=1}^{n} w_i \bar{r}_i$$

# Investment Process

Now for the portfolio with ′n′ number of assets, each has a certain random return and let it be denoted by $R_i$, $\forall$ i = 1, 2,……,n. Also suppose that out of the total money we have (be it our own or borrowed), we invest that total amount of money in the proportion of $w_i$ in the asset i, i.e., $w_1$ in asset 1, $w_2$ in asset 2 and so on till $w_n$ proportion is invested in asset n

# Investment Process

So we have $\sum_{i=1}^{n} w_i = 1$

This means that we have to invest all the money we have with us. Also remember that $1 \geq w_i \geq 0$, $\forall$ i = 1, 2,……,n. This is only possible iff we do not take the recourse of SS. If there is SS, $w_i$′s can be negative also. Remember that $w_i$′s are also called the weights in the respective assets.

# Investment Process

For the portfolio the total return (which is also random) is given by:

$$R_P = \sum_{i=1}^{n} w_i R_i$$

But we are interested in only the mean or expected return, which his given by:

$$E[R_P] = \overline{R}_P = \sum_{i=1}^{n} w_i \overline{R}_i$$

While the variance of the portfolio is given by:

$$V[R_P] = \sigma_P^2 = \sum \{R_P - \overline{R}_P\}^2 = \sum_{i,j=1}^{n} w_i w_j \operatorname{cov}(R_i, R_j) = \sum_{i,j=1}^{n} w_i w_j \rho_{i,j} \sigma_i \sigma_j$$

# Investment Process

This correlation coefficient $\rho_{i,j}$ between the $i^{th}$ and the $j^{th}$ asset can be positive, negative or zero. Hence we group assets, forming the portfolio is such a way that the correlation coefficient between the assets can be utilized to reduce the overall risk of the portfolio. This process by which we reduce the portfolio risk is known as diversification.

# Investment Process

Let us consider an example. Suppose there are two assets with $\bar{R}_1 = 0.12$ $\bar{R}_2 = 0.15$ $\sigma_1$=0.20, $\sigma_2$=0.18 and $\rho_{1,2}$=0.01. A portfolio is formed with weights $w_1$=0.25 and $w_2$=0.75.

Then $$\bar{R}_P = w_1\bar{R}_1 + w_2\bar{R}_2$$

$$\bar{R}_P = 0.25 * 0.12 + 0.75 * 0.15$$

$$\sigma_P^2 = w_1^2 * \sigma_1^2 + 2 * w_1 * w_2 * \rho_{1,2} * \sigma_1 * \sigma_2 + w_2^2 * \sigma_2^2$$

$$\sigma_P^2 = 0.25^2 * 0.20^2 + 2 * 0.25 * 0.75 * 0.01 * 0.20 * 0.18 + 0.75^2 * 0.18^2$$

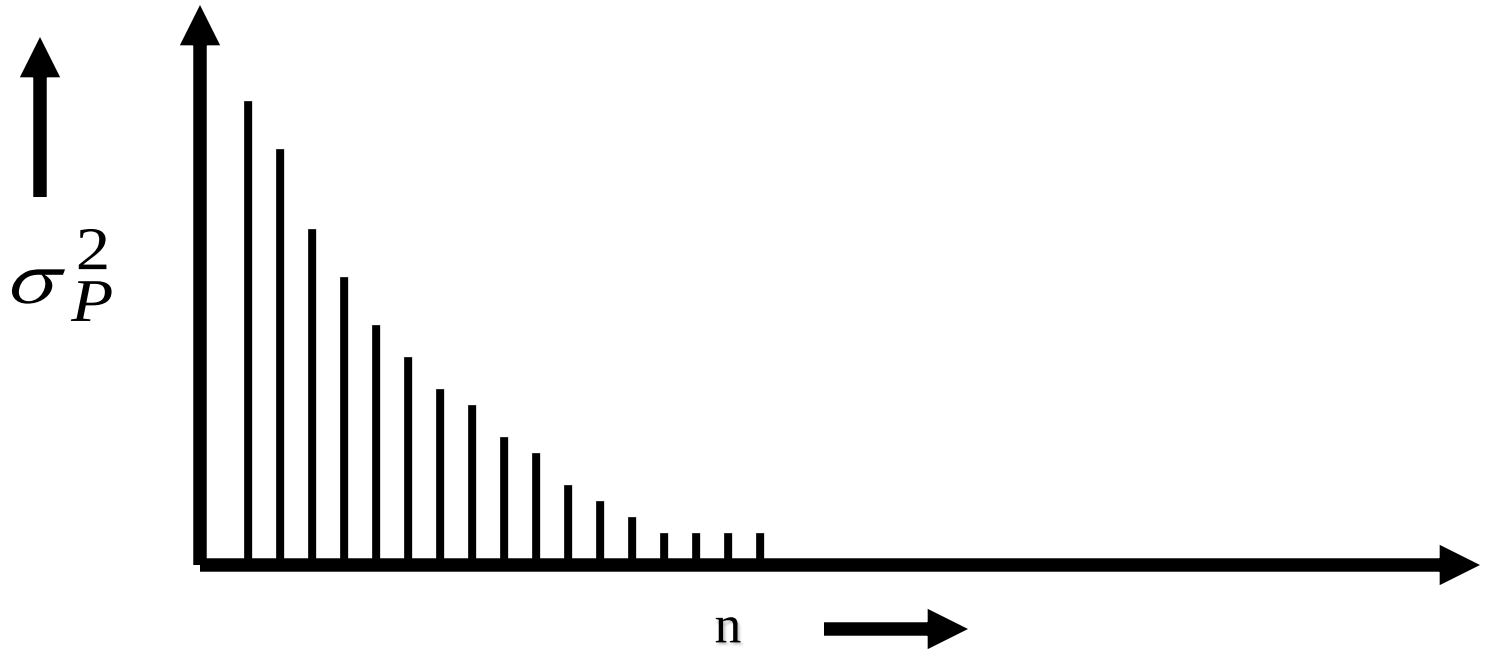
# Investment Process

For ease of understanding the concept of diversification let us consider a hypothetical example, where we have ′n′ number of assets, denoted by i =1, 2,….., n, such that

1) The prices of assets are moving in such a way that their respective prices are almost uncorrelated or the correlation is very low.
2) The return of each asset has an average value (mean) of ′m′.
3) The variance (risk) of each asset is $'\sigma^2'$.
4) Weights of each asset considered in the portfolio is assumed to be of equal proportion, i.e.,$w_i = 1/n$ for each i.

# Investment Process

$$\bar{R}_P = \frac{1}{n}\sum_{i=1}^{n} R_i = m \qquad \sigma_P^2 = \frac{1}{n^2}\sum_{i=1}^{n}\sigma^2 = \frac{\sigma^2}{n}$$



# Investment Process

Thus if we consider diversification, portfolios with only a few assets may be subject to a high degree of risk, represented by a relatively large variance. So the variance of the return of a portfolio can be reduced by including additional assets in the portfolio.

# Investment Process

Consider another example but now with correlated assets. The return and variance of the assets are as in the above example with mean m and variance $\sigma^2$, but now each pair of assets has a covariance of (for $i \neq j$) $0.3*\sigma_i*\sigma_j = 0.3*\sigma^2$.

Hence $$\sigma_P^2 = \frac{0.7*\sigma^2}{n} + 0.3*\sigma^2$$

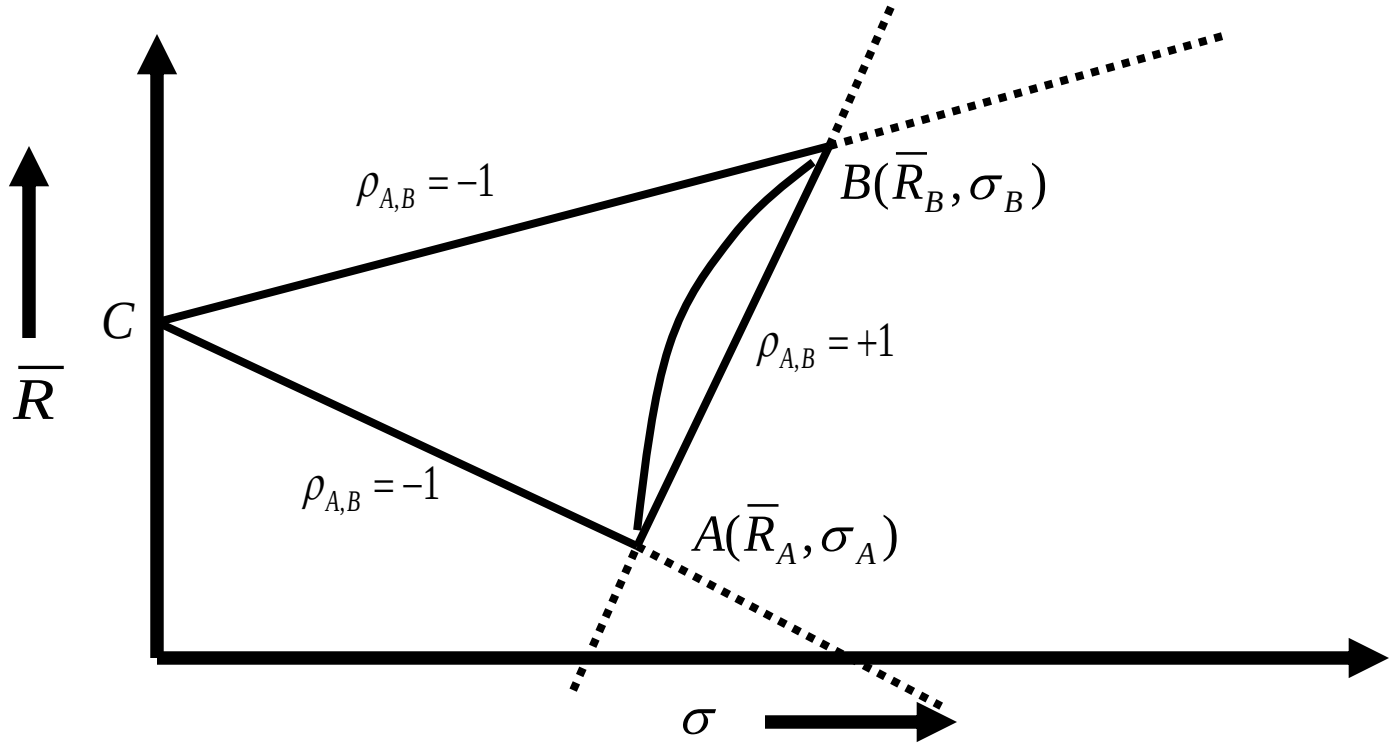
so as $n \to \infty$, we have the second term only. Thus $n \to \infty$, or diversification implies reduction in portfolio risk.

# Investment Process

For assets A and B the respective returns and variances are given. Let the corresponding weights be $w_1 = (1 - \alpha)$ and $w_2 = \alpha$, i.e., $w_1 + w_2 = 1$. As $\alpha$ varies from 0 to 1, the portfolio goes from one that contains only asset A to one that contains mixture of asset A and B and then to one that contains asset B. Remember values of $\alpha$ outside the range $0 \leq \alpha \leq 1$ make one or other of the weight as negative, corresponding to SS. The curve looks like that shown in the diagram above but what exact shape we get will depend on the value of correlation coefficient ($\rho_{A,B}$). The solid portion of the curve corresponds to positive combination of two assets, while the hashed portion corresponds to the shorting of assets. Remember one important thing. The curve must always lie inside the triangular region, denoted by ABC.

# Investment Process

Combination of **two** assets on the risk-return diagram or the mean-standard deviation diagram would give us the following:



# Investment Process

The curve in the risk-return diagram defined by non-negative mixtures of two assets A and B lies within the triangular region defined by the two original assets and the point on the vertical axis of height

$$\frac{\overline{R}_A\sigma_A + \overline{R}_B\sigma_B}{\sigma_A + \sigma_B}$$

Now if we denote the portfolio formed by A and B as P, then we have the return and variance for the portfolio about which we will now discuss.

# Investment Process

Thus:

$$\bar{R}_P(\alpha) = \sum_{i=1}^{2} w_i \bar{R}_i = (1-\alpha)\bar{R}_A + \alpha \bar{R}_B$$

$$\sigma_P^2(\alpha) = \sum_{i,j=1}^{n} w_i w_j \operatorname{cov}(R_i, R_j) = (1-\alpha)^2 \sigma_A^2 + 2\alpha(1-\alpha)\sigma_{A,B} + \alpha^2 \sigma_B^2$$

$$\sigma_P(\alpha) = \sqrt{(1-\alpha)^2 \sigma_A^2 + 2\rho_{A,B}\alpha(1-\alpha)\sigma_A\sigma_B + \alpha^2 \sigma_B^2}$$

# Investment Process

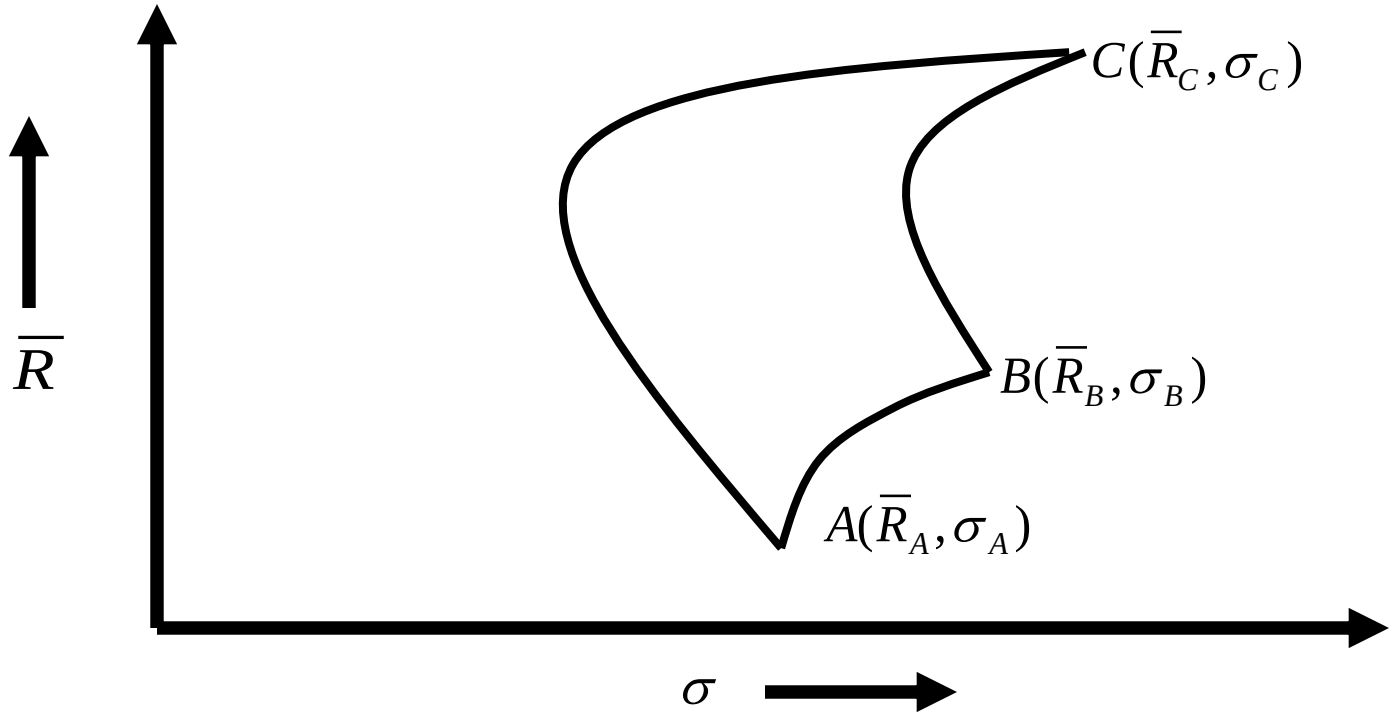
For $\rho$ = +1 and $\rho$ = -1, we have two different equations which are

$$\sigma_P(\alpha)^* = (1-\alpha)\sigma_1 + \alpha\sigma_2$$

$$\sigma_P(\alpha)_* = \left|(1-\alpha)\sigma_1 - \alpha\sigma_2\right|$$

# Investment Process

Combination of **three** assets on the risk-return diagram or the mean-standard deviation diagram would give us the following:

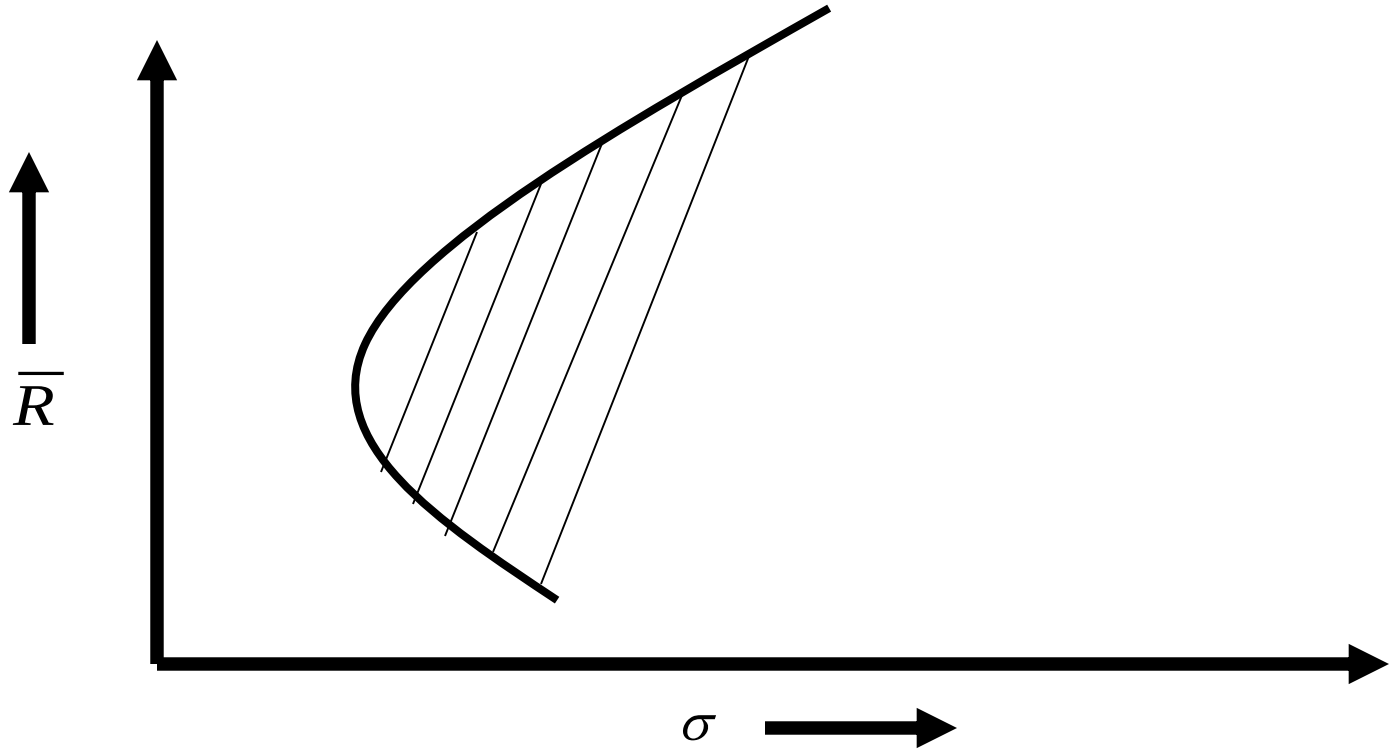


# Investment Process

Now suppose that in place of only two assets you have ′n′ such assets, where each has a certain return and variance and they also have some correlation between them. We can plot these assets as points on the risk-return diagram. We can form different combinations of portfolios from these ′n′ assets. Like taking two assets at a time (${}^{n}C_{2}$), taking three assets at a time (${}^{n}C_{3}$) and so on till we take all the ′n′ assets at the same time (${}^{n}C_{n}$). Each time we take a certain number of assets, we must ensure that the weights of all the assets taken for forming a particular portfolio should add up to 1. All these set of points that correspond to portfolios is called the **feasible set** or **feasible region**. **This feasible region is convex to the left**.

# Investment Process

Feasible set for ′n′ assets. By ′n′ we mean an infinite number of assets



# Investment Process

Few important concepts

- Risk averse person is one who would always seek such a portfolio from a group of portfolios for which the risk is minimum, given his/her return profile.
- Risk preferring person is one who would do the reverse, i.e., seek portfolio which has the highest risk. But remember in that case he/she would be compensated with a higher return from that portfolio.
- Risk neutral person is one who is indifferent between risks for a certain return of portfolios and would choose any one of the portfolios.
- Nonsatiation: It reflects the idea that everything being equal, investors always want more (money). Hence they want highest possible expected return for a given standard deviation, i.e., risk.

# Investment Process

An important thing to note is that we will be analyzing the portfolio choice for risk averse people with the added condition of nonsatiation. With this consideration we can easily see that only the upper part of the minimum variance set will be of interest for these investors, which is denoted in the slide. This curve is known as the efficient frontier. This frontier is the conglomeration of only efficient portfolios, in the sense that they provide the best mean-variance combinations for most investors.

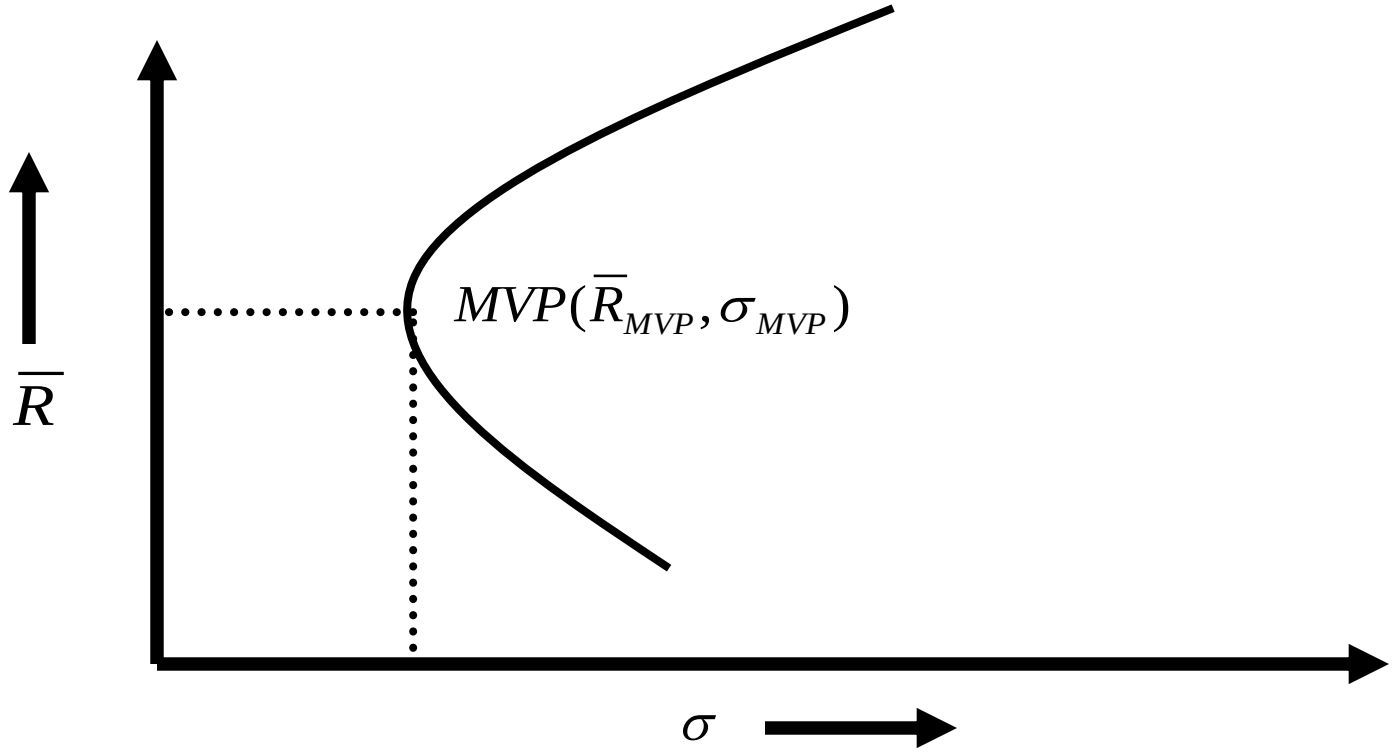
# Investment Process

**Minimum variance set**

The left boundary of a feasible set is called the minimum variance frontier, since for any value of the mean rate of return, the feasible point with the smallest variance (or standard deviation) is the corresponding left boundary point. There is a special point on this set having minimum variance and it is called the **minimum variance point**, which is denoted by MVP.

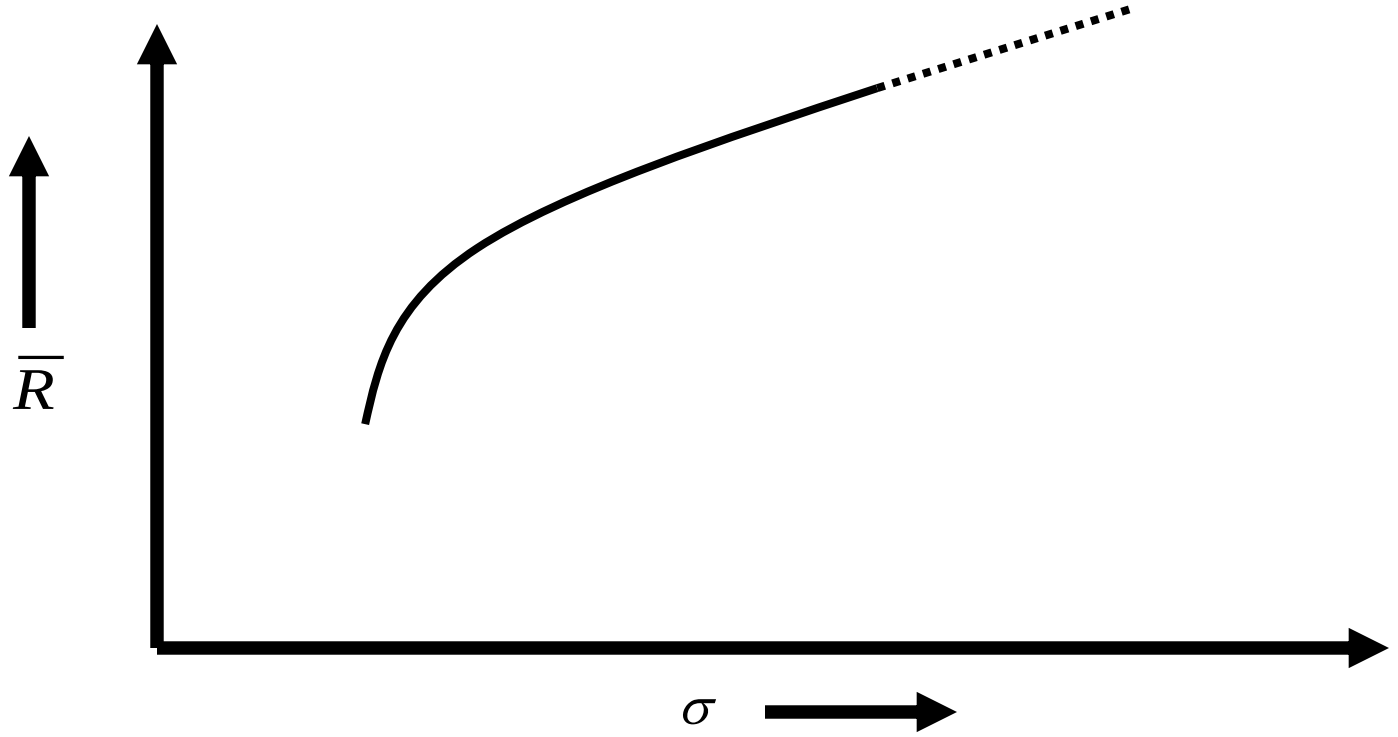
# Investment Process

**Minimum variance set and minimum variance point**



# Investment Process

**Efficient frontier**



# Markowitz Model

Assume the portfolio has ′n′ number of assets, each with a expected return of $\overline{R}_i$ and covariance of $\sigma_{i,j}$ $\forall$ i, j = 1, 2,……,n. Also suppose $w_i$ are the weights that sum up to 1. Then to find a minimum variance portfolio, we fix the mean portfolio return at some arbitrary value $\overline{R}^*$

# Investment Process

Thus

$$\min \frac{1}{2} \sum_{i,j=1}^{n} w_i w_j \sigma_{i,j}$$

subject to

$$\sum_{i=1}^{n} w_i \overline{R}_i = \overline{R}^*$$

$$\sum_{i=1}^{n} w_i = 1$$

$$w_i \geq 0$$

# Investment Process

In the previous optimization problem we have considered that there is no short selling, hence the weights have to be greater than 0.

# Investment Process

Now if we consider that SS is present, then the optimization problem becomes

# Investment Process

Thus

$$\min \frac{1}{2} \sum_{i,j=1}^{n} w_i w_j \sigma_{i,j}$$

subject to

$$\sum_{i=1}^{n} w_i \overline{R}_i = \overline{R}^*$$

$$\sum_{i=1}^{n} w_i = 1$$

$w_i$'s are unbounded

# Investment Process

The problem could also have been solved as (SS not considered)

$$\max \sum_{i=1}^{n} w_i \overline{R}_i$$

subject to

$$\sum_{i,j=1}^{n} w_i w_j \sigma_{i,j} = \sigma^{2*} \qquad \sum_{i=1}^{n} w_i = 1 \qquad w_i \geq 0$$

# Investment Process

The problem could also have been solved as (SS allowed)

$$\max \sum_{i=1}^{n} w_i \bar{R}_i$$

subject to

$$\sum_{i,j=1}^{n} w_i w_j \sigma_{i,j} = \sigma^{2*}$$

$$\sum_{i=1}^{n} w_i = 1$$

$w_i$'s unbounded

# Investment Process

After solving the optimization **minimization** problem (considering SS is allowed), we have

$$\sum_{j=1}^{n} \sigma_{i,j} w_j - \lambda \overline{R}^* - \mu = 0 \qquad \forall \text{ i = 1, 2,….., n}$$

$$\sum_{i=1}^{n} w_i \overline{R}_i = \overline{R}^*$$

$$\sum_{i=1}^{n} w_i = 1$$

# Investment Process

Suppose there are three (3) uncorrelated assets with variances of 1 and mean values of 1, 2 and 3 respectively. Solving the equations we have

$$w_1 = \frac{4}{3} - \frac{\bar{R}^*}{2} \qquad w_2 = \frac{1}{3} \qquad w_3 = \frac{\bar{R}^*}{2} - \frac{2}{3}$$

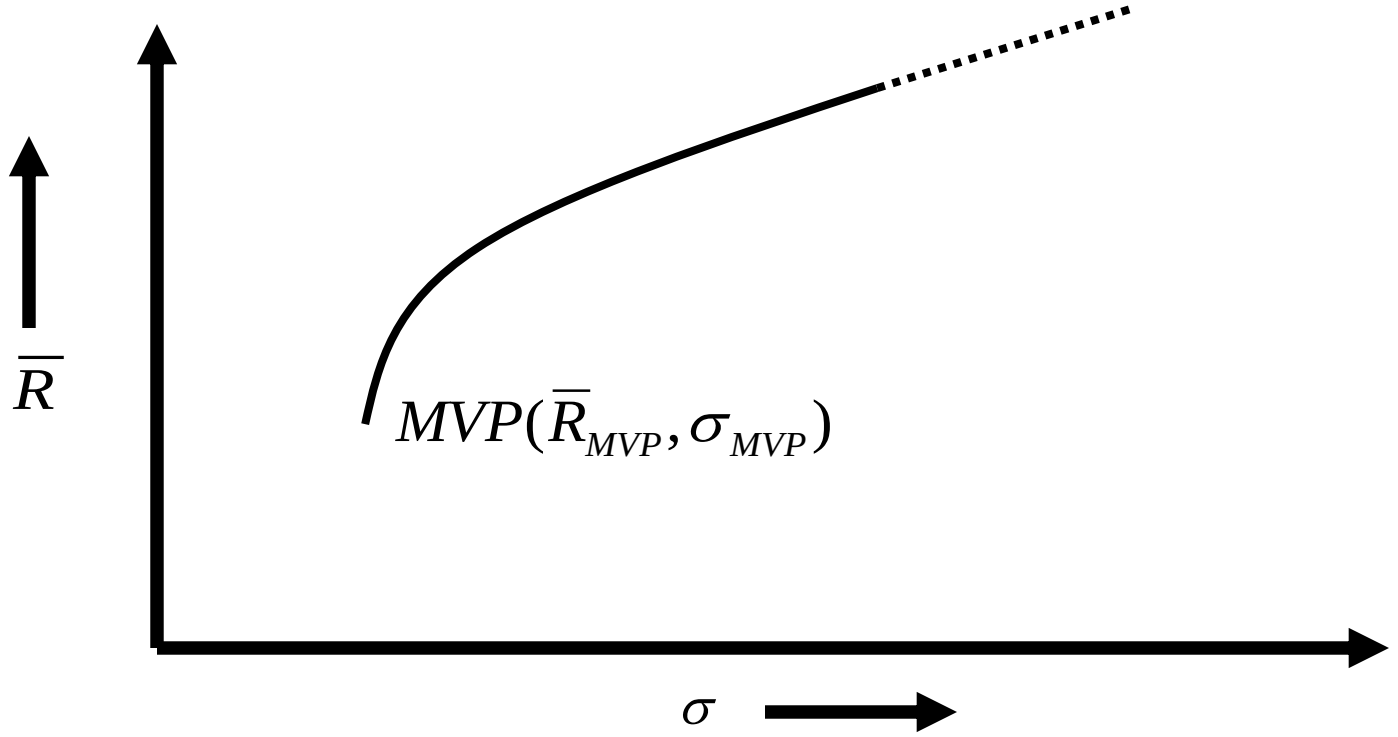
# Investment Process

Moreover the mean and risk (standard deviation) for the portfolio thus formed are:

$$\overline{R}_{P(n=3)} = \overline{R}^{*}$$

$$\sigma_{P(n=3)} = \sqrt{\frac{7}{3} - 2 * \overline{R}^{*} + \frac{\overline{R}^{*2}}{2}}$$

# Investment Process



# Investment Process

Suppose we have two portfolios (after solving the optimization problem) corresponding to two different starting values $\overline{R}_1^*$ and $\overline{R}_2^*$

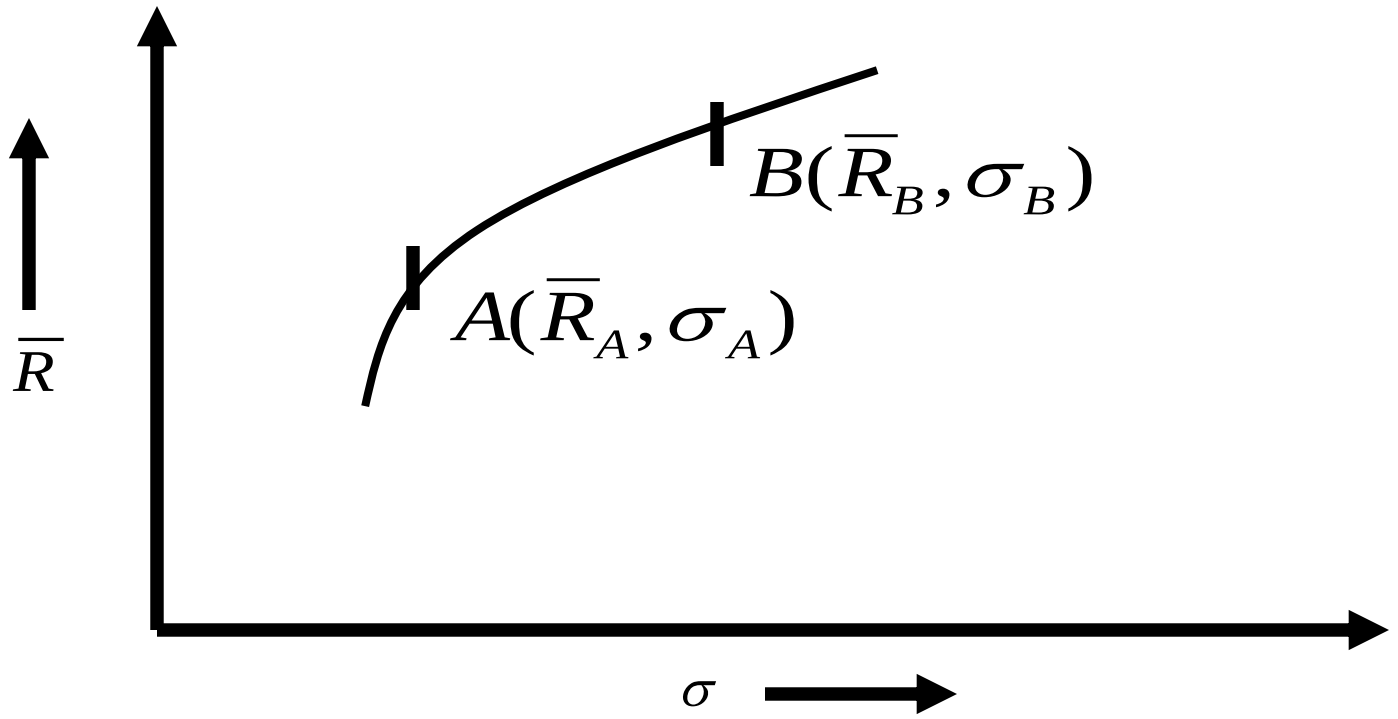
Now combining them in some proportions such that the respective weights adds up to 1 we can have an infinite number of combinations, each of which is a solution to the optimization problem. This gives us the idea of the two fund theorem.

# Investment Process

## Two fund theorem

Two efficient funds (portfolios) can be established so that any efficient portfolio can be duplicated in terms of mean and variance, as a combination of these. In other words all investors seeking efficient portfolios need only invest in a combinations of these portfolios.

# Investment Process



# Investment Process

The optimization problem under the consideration that no SS is allowed was as follows

$$\min \frac{1}{2} \sum_{i,j=1}^{n} w_i w_j \sigma_{i,j}$$

subject to

$$\sum_{i=1}^{n} w_i \bar{R}_i = \bar{R}^* \qquad \sum_{i=1}^{n} w_i = 1 \qquad w_i \geq 0$$

# Investment Process

This is a non-linear optimization problem and cannot be solved as shown before for the case when SS is allowed.

# Investment Process

We have the following data (variance covariance and returns) which is given below

| Security # | Variance-Covariance Values | | | | | Returns |
|---|---|---|---|---|---|---|
| 1 | 2.30 | 0.93 | 0.62 | 0.74 | -0.23 | 15.10 |
| 2 | 0.93 | 1.40 | 0.22 | 0.56 | 0.26 | 12.50 |
| 3 | 0.62 | 0.22 | 1.80 | 0.78 | 0.27 | 14.70 |
| 4 | 0.74 | 0.56 | 0.78 | 3.40 | -0.56 | 9.02 |
| 5 | -0.23 | 0.26 | -0.27 | -0.56 | 2.60 | 17.68 |

# Investment Process

For solving consider two sets of solution:

- $\lambda$=0 and $\mu$=1 (gives MVP)
- $\lambda$=1 and $\mu$=0

# Investment Process

For $\lambda$=0 and $\mu$=1 we solve $\sum_{j=1}^{5}\sigma_{i,j}v_j^1 = 1$

solving we have $\mathbf{v^1}$=(0.141,0.401,0.452,0.166,0.440).

Normalizing the $v^1_i$'s so that the sum is 1 we obtain the weights $w_i^1 = \frac{v_i^1}{\sum_{j=1}^{5} v_j^1}$

$\mathbf{w^1}$=(0.088,0.251,0.282,0.104,0.275)

Thus we finally have $\overline{R}^1 = 14.413 \quad \sigma^1 = 0.791$

# Investment Process

For $\lambda$=1 and $\mu$=0 we solve $\sum_{j=1}^{5}\sigma_{i,j}v_j^2 = \overline{R}_i$

solving we have $\mathbf{v^2}$=(3.652,3.583,7.248,0.874,7.706)

Normalizing the $v^2_i$'s so that the sum is 1 we obtain the weights $w_i^2 = \frac{v_i^2}{\sum_{j=1}^{5} v_j^2}$

$\mathbf{w^2}$=(0.158,0.155,0.314,0.038,0.334)

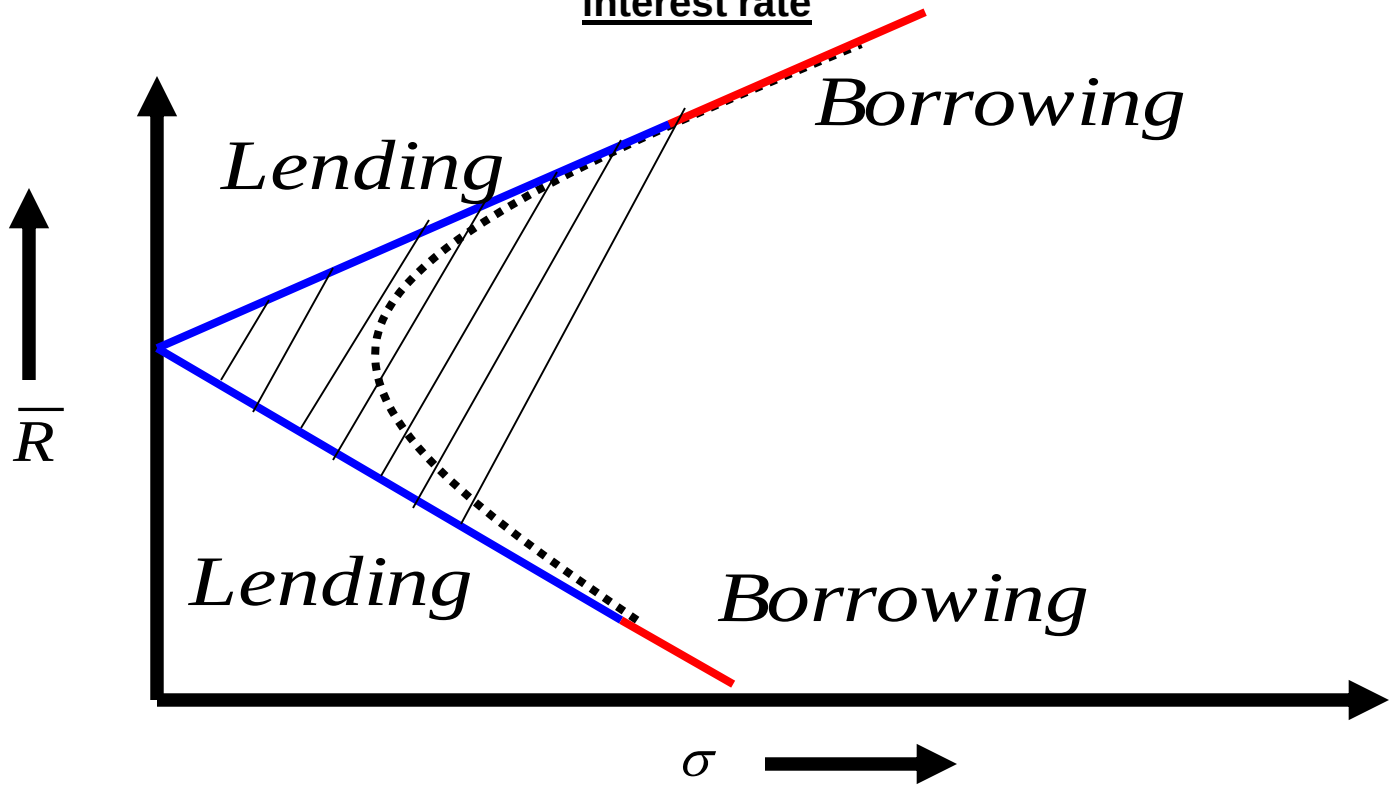
Thus we finally have $\overline{R}^2 = 15.202 \quad \sigma^2 = 0.812$

# Investment Process

As already explained we are aware of the concept of risk free interest rate ($R_f$). The question we are interested in knowing immediately is how does the feasible set, minimum variance set and efficient frontier look like when we include the risk free interest rate along with n risky assets.

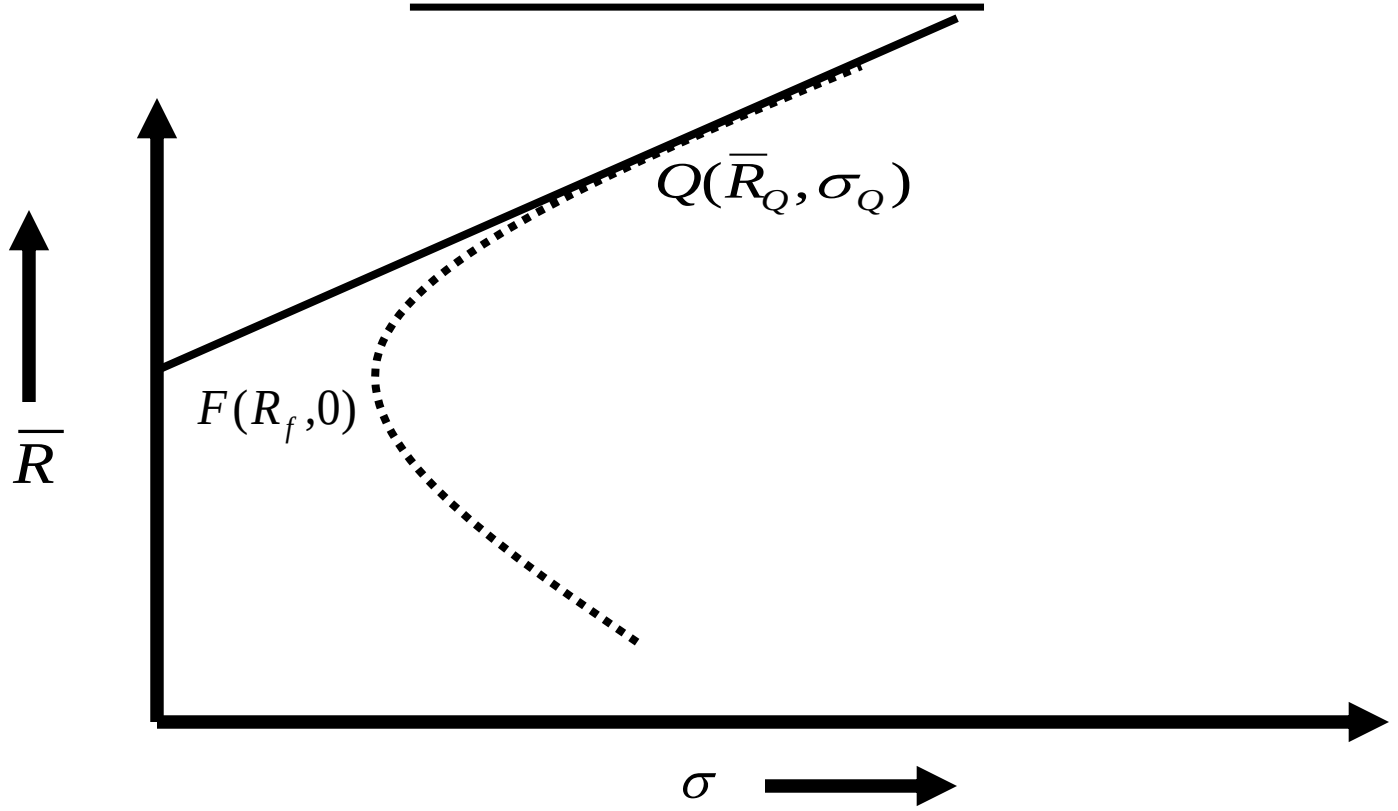
# Investment Process

**Feasible set, minimum variance set and efficient frontier considering risk free interest rate**



# Investment Process

**One fund theorem**



# Investment Process

## One fund theorem

There is a single fund Q of risky assets such that any efficient portfolio can be constructed as a combination of the fund Q and the risk free interest rate F.

# Investment Process

**Problem # 1**

The correlation coefficient between assets A and B is 0.1 and their returns and standard deviations are as follows

| Asset # | SD | Return |
|---|---|---|
| A | 10.0% | 15.0% |
| B | 18.0% | 30.0% |

Find

- The proportion of A and of B that define a portfolio (consisting of A and B only) having minimum standard deviation
- What is the value of minimum SD for the portfolio
- What is the corresponding mean return of the portfolio thus formed at the minimum SD point

# Investment Process

**Problem # 1 (alternative example)**

Take two scripts X and Y (from the five examples you as a group have chosen). For this example take the time period from 01-07-2005 to 31-12-2005. Find

- The proportion of X and of Y that define a portfolio (consisting of X and Y only) having minimum standard deviation
- What is the value of minimum SD for the portfolio formed with X and Y only.
- What is the corresponding mean return of the portfolio thus formed at the minimum SD point

# Investment Process

**Problem # 2**

There are three (3) assets and their covariances and returns are given below

| Asset # | Variance Covariance | | | Return |
|---|---|---|---|---|
| 1 | 2 | 1 | 0 | 0.4 |
| 2 | 1 | 2 | 1 | 0.8 |
| 3 | 0 | 1 | 2 | 0.8 |

Questions

- Find the MVP
- Find another efficient portfolio
- Draw the efficient frontier considering correlation coefficient between the two funds is 0.5
- If risk free interest rate is 0.2, find the efficient portfolio of risky assets
- Draw the efficient frontier when there is risk free interest rate

# Investment Process

**Problem # 2 (alternative example)**

There are three (3) scripts (from the five examples you as a group have chosen). For this example take the time period from 01-01-2006 to 30-06-2006. Find

- The MVP.
- Find another efficient portfolio.
- Draw the efficient frontier considering correlation coefficient as calculated by you in this time period.
- Take the value of risk free interest rate as found by you from the data and then find the efficient portfolio of risky assets.
- Draw the efficient frontier when there is risk free interest rate.

# Investment Process

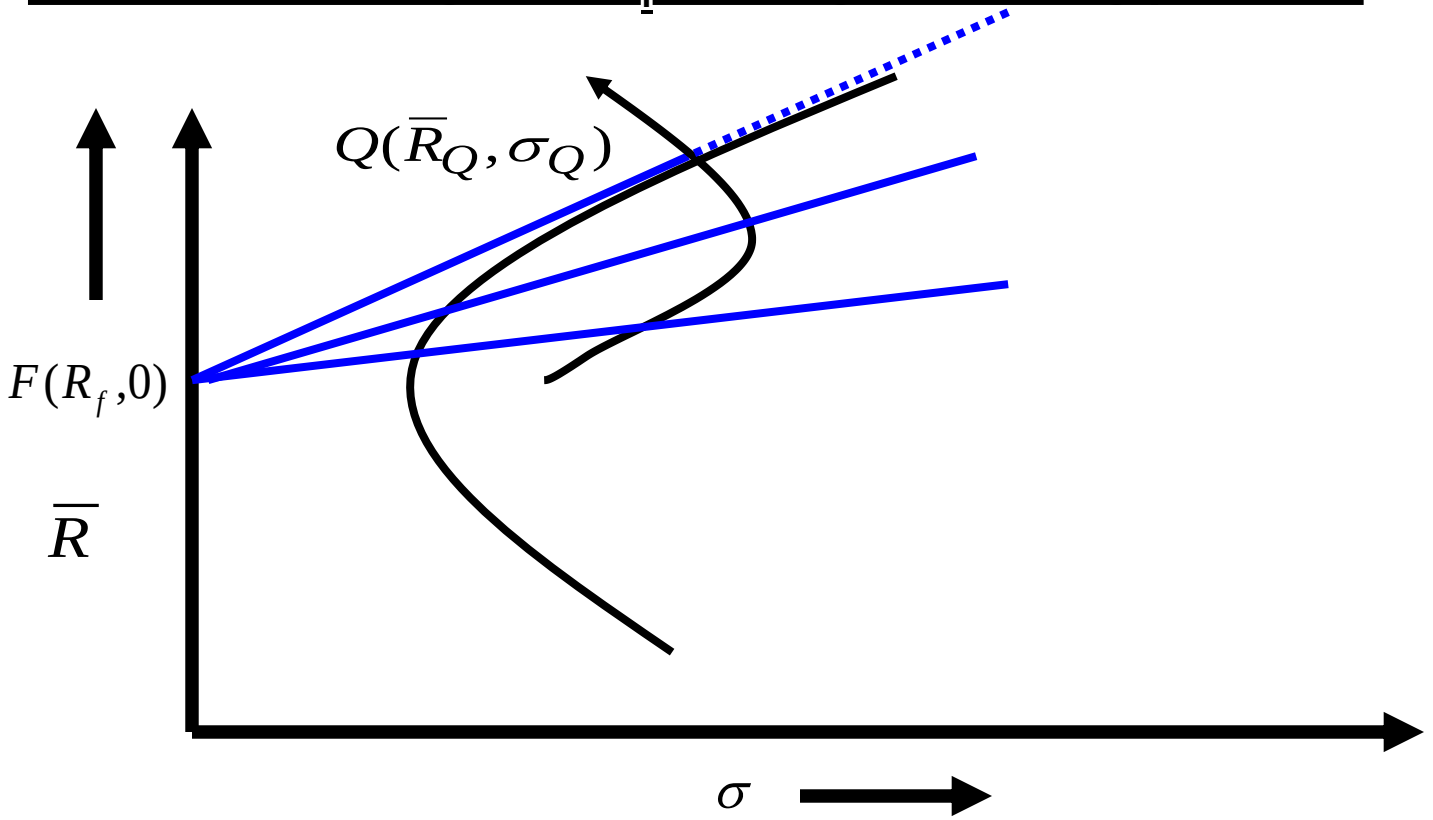
**Techniques for calculating the efficient frontier**

We will discuss here the following

- SS allowed along with $R_f$ lending/borrowing possible
- SS allowed but $R_f$ lending/borrowing not permitted
- SS disallowed along with $R_f$ lending/borrowing possible
- SS disallowed nor is $R_f$ lending/borrowing allowed
- Incorporation of other constraints/assumptions

# Investment Process

**SS allowed along with $R_f$ lending/borrowing possible**



# Investment Process

For SS allowed along with $R_f$ lending/borrowing possible we should have the ray such that it is furthest in the **counter clockwise** direction. The **efficient frontier** is the entire length of the ray extending from F to Q and beyond. Remember the loci F-Q is tangent to the point Q. Different point on the line represent different amounts of risk less lending and/or borrowing in combination with the optimum portfolio of risky assets.

# Investment Process

Thus we have:

$$\max \theta = \frac{\bar{R}_Q - R_f}{\sigma_Q}$$

such that $\sum_{i=1}^{n} w_i = 1$

There are two solution methods and one is using Lagrangian. We will discuss that only

# Investment Process

Thus we have:

$$\max \theta = \frac{\sum_{i=1}^{n} w_i (\bar{R}_i - R_f)}{(\sum_{i=1}^{n} w_i w_j \sigma_{i.j})^{1/2}}$$

such that $$\sum_{i=1}^{n} w_i = 1$$

# Investment Process

The solution is as given below (for k = 1, 2, ….., n)

$$\sum_{j=1}^{n} w_j' \sigma_{k,j} = \bar{R}_k - R_f$$

$$w_i = \frac{w_i'}{\sum_{j=1}^{n} w_j'}$$

# Investment Process

**Problem # 3**

There are three (3) assets and their covariances and returns are given below. Also the risk free interest rate is 5

| Asset # | Variance | Covariance | | Return |
|---|---|---|---|---|
| 1 | 36 | 9 | 36 | 14 |
| 2 | 9 | 9 | 9 | 8 |
| 3 | 36 | 9 | 225 | 20 |

Find the optimum portfolio thus formed with these assets

# Investment Process

Using the solution methodology we have

$$\bar{R}_1 - R_f = w_1' \sigma_1^2 + w_2' \sigma_{12} + w_3' \sigma_{13}$$

$$\bar{R}_2 - R_f = w_1' \sigma_{21} + w_2' \sigma_2^2 + w_3' \sigma_{23}$$

$$\bar{R}_3 - R_f = w_1' \sigma_{31} + w_2' \sigma_{32} + w_3' \sigma_3^2$$

# Investment Process

Thus

$14 - 5 = w'_1 * 36 + w'_2 * 9 + w'_2 * 36$

$8 - 5 = w'_1 * 9 + w'_2 * 9 + w'_3 * 9$

$20 - 5 = w'_1 * 36 + w'_2 * 9 + w'_3 * 225$

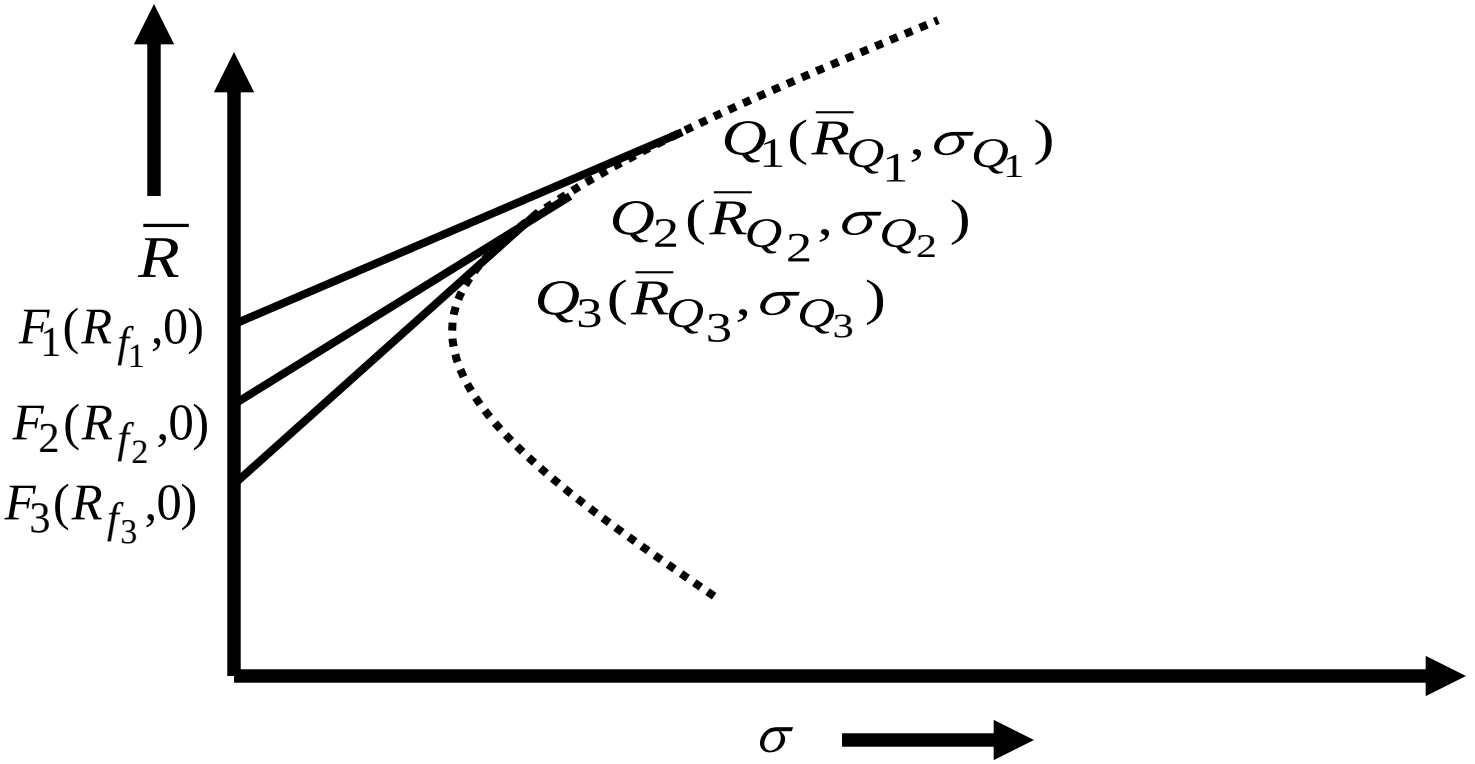
Finally: $w'_1 = 14/63$; $w'_2 = 1/63$; $w'_3 = 3/63$

$w_1 = 14/18$; $w_2 = 1/18$; $w_3 = 3/18$

Note as a cross validation you can check that the weights add up to 1.

# Investment Process

**SS allowed along but $R_f$ lending/borrowing not permitted**



# Investment Process

Assume that a certain risk less interest rate for lending/borrowing exits and using that particular interest rate we find the corresponding optimum portfolio and thus the corresponding point Q. Continue doing that till we are able to draw the entire portfolio shown by the dotted curve

# Investment Process

**SS disallowed along with $R_f$ lending/borrowing possible**

Thus we have $$\max \theta = \frac{\bar{R}_Q - R_f}{\sigma_Q}$$

such that $$\sum_{i=1}^{n} w_i = 1 \qquad w_i \geq 0$$

For this we have to use quadratic formulation and we would not discuss it here.

# Investment Process

**SS disallowed nor is $R_f$ lending/borrowing allowed**

Thus we have $$\min\{ \sum_{i=1}^{n} w_i^2 \sigma_{i,i} + \sum_{i \neq j=1}^{n} w_i w_j \sigma_i \sigma_j \}$$

such that $$\sum_{i=1}^{n} w_i \overline{R}_i = \overline{R}_P^*$$

$$\sum_{i=1}^{n} w_i = 1$$

$$w_i \geq 0$$

# Investment Process

Recall that we specify the return from the portfolio at some level and minimize the portfolio risk, which gives us a point on the efficient frontier. This is a simple solution which we have already discussed. Now varying the specified portfolio return from MVP to some point we get different values of the optimum portfolio on the efficient frontier, thus helping us to trace the efficient frontier